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One-dimensional vibratory instability of planar flames propagating in tubes

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A complete analysis of the one-dimensional vibratory instability of planar flames of premixed gases propagating in tubes is provided. The driving mechanism results from unsteady coupling between flame structure and acoustic waves through temperature fluctuations. In certain conditions, the strength of such an instability will be proved to be sufficiently strong to produce large-amplitude fluctuations as soon as the flame has travelled a distance of the order of the acoustic wavelength. Stability limits and total amplification of an initial perturbation are computed in the framework of the simple flame mode of a one-step exothermic reaction governed by an Arrhenius law with an activation energy much larger than the thermal energy. Diffusive and thermal effects within the flame are included with a Lewis number different from unity. Damping mechanisms associated with viscous and thermal dissipation at the walls, as well as with loss of acoustic energy by sound radiation from the open end of the tube, are retained. In ordinary conditions, for a reactive mixture with an effective Lewis number close to unity, the predicted instability is weak. In the framework of the simplified flame model used here, islands of strong instabilities are predicted to occur at low Mach numbers for Lewis numbers larger than unity.

1. Introduction

In one of the earliest photographic studies of flames, Mallard & Le Chatelier (1883) observed that a premixed gas flame propagating from the open to the closed end of a tube started to oscillate. This oscillation is associated with a spontaneous increase of the acoustic energy in the tube. The acoustic vibration is longitudinal, with a frequency close to an eigenmode of the tube. In general, the corresponding acoustic time has the same order of magnitude as the transit time of the flame. The details of the process depend on the nature and the equivalence ratio of the mixture and on the dimensions of the tube. This suggests the existence of a coupling mechanism between the flame structure and acoustic modes of the tube. Such a phenomenon is of fundamental importance since it shows how compressibility effects can spontaneously develop from an initially isobaric structure, the deflagration wave. Thus it may play a role in the process of transition to compressible combustion waves such as detonations, or in knocking in engines, where strong pressure disturbances develop.

This vibratory instability is related to the more general field of thermoacoustic instabilities which appear in systems in which a localized heat source interacts with acoustic vibrations. A general criterion for the appearance of such instabilities was developed very early by Rayleigh (1878): when heat is released locally and periodically in a gaseous medium, an acoustic oscillation is amplified if the oscillating components of pressure and heat flux are in phase. For reviews on this subject, see Putnam (1964) and Strehlow (1979).

The determination of the stability limits of the vibratory instability of flames remains an open question since the driving mechanism which is at the origin of the variations of the heat flux released by the freely propagating flame is still unclear. Up to now, three kinds of explanations have been proposed. Dunlap (1950) suggests that the vibrations could be maintained by the effect of pressure and temperature of incoming acoustic waves on the burning velocity. Kaskan (1953) proposes that when the flame edge penetrates into the thin acoustic boundary layer, flame area and thus heat flux fluctuate in the boundary layer since the flame front, disk shaped and planar in the centre part of the tube, oscillates around its mean position while the edge remains at rest. Markstein (1964) assumes that heat flux variations are due to variations of flame area by an acceleration instability, this last mechanism being certainly present when flames are cellular.

When planar propagation is considered, as is the case in Kaskan's (1953) experiments, only the first two mechanisms need be considered. The first mechanism is usually rejected since the effect of pressure variation on burning velocity is very weak. Such an argument is not fully convincing for two reasons. First of all, a detailed computation of the transfer function taking into account the flame structure to relate the jump of the acoustic velocity field across the flame to the pressure fluctuation, is still lacking. Thus, a correct evaluation of the magnitude of the corresponding effect has not yet been carried out. Secondly, damping effects which dissipate the acoustic energy, i.e. losses by sound emission at the open end of the tube and by viscous friction or by heat conduction to the tube walls are very weak too. Thus it is difficult to say which effect (destabilizing or stabilizing) will dominate without a detailed analysis.

The scope of this paper is to analyse in detail the stability limits of the vibratory instability for planar flames propagating in tubes, when the origin of heat flux variations comes from time-dependent modifications of the flame structure which are induced by acoustic waves. It will be shown that such an effect may be quantitatively sufficient to overcome the damping effects. Moreover, when the instability develops, its typical growth rate will be proved in some cases to be sufficiently strong to produce large-amplitude vibrations as soon as the flame has travelled a distance of the order of an acoustic wavelength. Thus, besides the well-known hydrodynamical and diffusive instabilities considered by Pelcé and Clavin (1982), a new type of intrinsic instability for planar flames propagating in tubes is pointed out and studied by the present analysis.

In the following section of the paper, the physical concepts used in the analysis are presented with a rough analysis based only on order of magnitude estimates. The model and the method which are used to solve the problem are presented in a third section. In a fourth section, an analysis of the response of the flame structure to an acoustic wave is carried out for a finite value of the gas expansion parameter. This result extends the one obtained previously by Van Harten, Kapila & Matkowsky (1984) in the limiting case of a negligibly small gas expansion parameter. Such an approximation cannot be used here because it corresponds to a limit where the driving mechanism of the vibratory instability vanishes. Then, the corresponding transfer function of the freely propagating flame is obtained in the same spirit as in the cases of a flame anchored to a grid (MacIntosh 1986) or of the Rijke tube where a hot grid is placed in a uniform flow (Nicoli & Pelcé 1989). The acoustic problem is solved in a fifth section by using the above-mentioned transfer function. The stability limits of the vibratory instability are first investigated for a flame which is assumed to be kept at a fixed position in the tube. The corresponding linear growth rate is obtained in the unstable domains.

2. Physical considerations

The Rayleigh criterion for instability (Rayleigh 1878) is easily derived in the framework of a simple model for a non-steady one-dimensional flow in a tube in which the heat release is concentrated in a region much smaller than the acoustic wavelength (Strehlow 1979). In such a case, the region of heat release may be considered as a discontinuity of the flow velocity. When damping mechanisms (viscosity, etc.) are neglected, the time derivative of the total energy per unit cross-sectional area, $\mathscr{E}$, stored in the acoustic modes of the tube is related to the jump of the velocity fluctuation δu across this thin zone:

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \delta p(\delta u_2 - \delta u_1). \tag{2.1}$$

Here δp is the pressure fluctuation, δu_2 and δu_1 the velocity fluctuations at the discontinuity on the downstream and upstream sides respectively. The jump $(\delta u_2 - \delta u_1)$ is a quantity that fluctuates with the acoustic mode of the tube. Because of the gas expansion, this jump is, in most cases, proportional to the fluctuation of the rate of heat release per unit cross-sectional area, δq, with a positive dimensionless coefficient. In such a case (2.1) shows that the instability appears ($\mathrm{d}\mathscr{E}/\mathrm{d}t > 0$) as soon as δq is in phase with δp, i.e. when the local addition is positively correlated with the available pressure fluctuation. The main problem in each particular case is to determine the relation linking δq to δp or more directly $(\delta u_2 - \delta u_1)$ to δp. The so-called transfer function characterizes the response of the inner structure of the heat release region to an acoustic perturbation. The purpose of this section is to discuss the orders of magnitude of these phenomena in the case where the thin zone of heat release is a premixed gas flame.

In the isobaric approximation, the structure of a planar premixed flame is governed by a system of reaction–diffusion equations for the temperature and species mass fraction. The main parameters characterizing the steady flame structure are the flame temperature T_2, the laminar flame speed U_L and the transit time τ_t. The flame thickness is defined by $d = \tau_\mathrm{t} U_\mathrm{L}$ and one has the following order of magnitude estimates:

$$U_\mathrm{L} \approx \left(\frac{D_2}{\tau_\mathrm{t}}\right)^{\frac{1}{2}}, \quad d \approx (D_2 \tau_\mathrm{t})^{\frac{1}{2}} \quad \text{with} \quad \tau_\mathrm{t} = \frac{d}{U_\mathrm{L}}, \tag{2.2}$$

and where D_2 is the thermal diffusivity of the gaseous mixture at temperature T_2. The heat flux q released in this steady regime of combustion is $q = \rho_1 U_\mathrm{L} C_p(T_2 - T_1)$, where C_p is the specific heat of the gaseous mixture, and T_1 and ρ_1 are the temperature and the density of the fresh mixture. The study of the flame structure provides us with the expression of the transit time τ_t and, thus, of U_L, d and q in terms of the physico-chemical properties of the reactive mixture, which are mainly characterized by the typical diffusion coefficient, D_2, and the reaction time τ_r of the exothermic reaction defined at the burned gas temperature T_2. This characteristic time is related to the microscopic collision time τ_c at the same conditions through an Arrhenius factor,

$$\tau_\mathrm{r} = \tau_\mathrm{c} \exp(E/\mathscr{R}T_2), \tag{2.3}$$

where E is the activation energy of the reaction. The essential characteristic of a combustion process is the large value of the activation energy compared to the thermal one: $E/\mathscr{R}T_2 \gg 1$. As a consequence the laminar flame speed and, thus, the heat flux are very sensitive to temperature variations. This sensitivity is characterized by a large dimensionless number $\beta \gg 1$, called the Zel'dovich number. A small relative variation of the flame temperature induces a modification of U_L and q of order unity:

$$\frac{\delta T_2}{T_2} = O\left(\frac{1}{\beta}\right) \Rightarrow \frac{\delta U_L}{U_L} = O(1) \quad \text{and} \quad \frac{\delta q}{q} = O(1). \tag{2.4a}$$

Typical orders of magnitude are:

$$\beta = 10, \quad \tau_t \approx 10^{-4}\,\text{s}, \quad U_L \approx 30\,\text{cm/s}, \quad d \approx 3\times 10^{-2}\,\text{cm}, \quad M \approx 10^{-3}, \tag{2.4b}$$

where M is the Mach number based on the flame speed U_L and a characteristic sound speed c_1 of the gaseous fresh mixture, $M = U_L/c_1$.

The oscillatory instability of planar flames propagating in tubes was studied experimentally by Kaskan (1953) who attributed the instability mechanism to the heat flux variation inside the viscous layer of thickness h at the edge of the flame front. When the Prandtl number is of order unity, the order of magnitude of h may be written as $h \approx (D_2 \tau_a)^{\frac{1}{2}}$, where $1/\tau_a$ is the acoustic frequency. When τ_a is sufficiently small, viscous effects are located in a thin boundary layer, $h \ll R$, where R is the radius of the tube. This is the case of the experiments of Kaskan in which the flame front remains planar everywhere except in the boundary layer. The local two-dimensional character of the boundary layer prevents a detailed analysis of the variation of the heat flux at the edge of the flame. Moreover, how such a local phenomenon could influence the bulk of the flow to be coupled with acoustic oscillatory modes as described by (2.1) is not explained. Kaskan postulated that the heat flux variation is associated with the modification of the area of the flame inside the boundary layer. When the dead space between the flame and the wall is larger than the thickness of the viscous boundary layer the corresponding heat flux variation disappears and Kaskan concludes that the instability mechanism dies out. As the dead space is of the same order of magnitude as the flame thickness d, the approximative instability criterion proposed by Kaskan may be written as $\tau_a < \tau_t$. We shall show here that the coupling between the planar flame structure and the longitudinal acoustic modes produces another instability mechanism which, *a priori*, cannot be neglected in particular at high frequency in the domain where the mechanism described by Kaskan disappears.

In a case of a flame anchored to a grid, the flame temperature T_2 depends on both the heat of reaction and the heat losses to the grid. This last mechanism depends on the position of the reaction zone relative to the fixed grid. The fluctuations of flow velocity associated with the acoustic wave induce an oscillation in the position of the reaction zone and thus a fluctuation of T_2. The resulting fluctuation in the heat release leads to a strong instability, studied by Bailey (1957). The corresponding transfer function has been computed more recently by McIntosh (1986).

Neither of the two above-mentioned mechanisms can be invoked for planar flames propagating freely in a tube when the dead space is larger than the thickness of the viscous boundary layer. The problem is described by one-dimensional reaction–diffusion equations which are translationally invariant and which do not include any inertial effects in the response of the flame to homogeneous fluctuations of the velocity field. Thus, the interaction between such a flame structure and acoustic modes cannot be treated in the strict isobaric approximation. One has to consider the pressure effects appearing in the conservation equation of enthalpy. The relative variation of temperature and pressure in an acoustic mode are of the same order of magnitude:

$$\frac{\delta T}{T} \approx \frac{\delta p}{p} \approx \frac{\rho c \delta u}{p} = M\left(\frac{\delta u}{U_L}\right), \tag{2.5}$$

where δu is the typical amplitude of the velocity fluctuation of the acoustic mode. According to (2.4*a*), this temperature fluctuation induces a relative modification of the flame structure and of the heat release of the order $\beta M(\delta u/U_L)$. For a relative gas expansion across the flame or order unity, one may expect the same order of magnitude for the relative velocity jump across the flame to give:

$$\frac{\delta u_2 - \delta u_1}{U_L} \approx \beta M\left(\frac{\delta u}{U_L}\right). \tag{2.6}$$

The energy per unit cross-sectional area, $\mathscr{E}$, which is stored in the acoustic mode of the tube is of order $(\tau_a \, \delta u \, \delta p)$. Thus, in the case of an unstable configuration, the characteristic order of magnitude of the growth rate of such an acoustic instability, $1/\tau_{ins} \equiv \mathscr{E}^{-1} \partial \mathscr{E}/\partial t$, would be, according to (2.1) and (2.6):

$$1/\tau_{ins} \approx (\beta M)/\tau_a, \tag{2.7}$$

where because of the very subsonic character of the flame propagation, the dimensionless coefficient βM is a small number ($\beta M \approx 10^{-2}$, see (2.4*b*)). Nevertheless the cumulative effect when the flame propagates in a tube could be very strong. For example if the instability criterion concerns lengths in the tube which are of the same order as the acoustic wavelength $\Lambda \approx c_1 \tau_a$, the residence time t_{res} which is defined as the time spent by the flame in the unstable region, is very long compared with the acoustic time: $t_{res} \equiv \Lambda/U_L \approx \tau_a/M$. Thus, the very small number M appearing in (2.7) is cancelled in the total growth rate to give: $t_{res}/\tau_{ins} \approx \beta$. As a consequence, a perturbation of the flame position with an initial amplitude A_i is expected to reach a final amplitude A_f which is increased by a very large factor $\exp \beta$,

$$A_f/A_i = \exp \beta. \tag{2.8}$$

Nevertheless, because of the smallness of the factor βM in the right-hand side of (2.7), one must make sure that the damping mechanisms do not dominate the instability mechanism. A precise analysis of this topic will determine the stability limits of the flame propagating in a tube presented in §5. Here, we shall consider only the order of magnitude of these phenomena. According to the presentation by Landau & Lifshitz (1959) of Kirchhoff's results concerning the acoustic damping associated with viscosity and heat transfer at the wall, the corresponding damping rate may be evaluated as follows. For a Prandtl number of order unity, the energy dissipated in the boundary layer per unit time and per unit tube is of order of $\rho D_2 (\delta u)^2 R/h$ with $h = (D_2 \tau_a)^{\frac{1}{2}}$. The acoustic energy which is stored in the acoustic mode may be written per unit tube length as $\rho(\delta u)^2 R^2$. The corresponding damping rate is

$$1/\tau_{dam} \approx (D_2/R^2\tau_a)^{\frac{1}{2}}.$$

By rewriting the right-hand side of the above equation as $(D_2 \tau_a/\Lambda^2)^{\frac{1}{2}} (\Lambda/R)/\tau_a$ and using $\Lambda \approx c_1 \tau_a$ with (2.2) written in the form $D_2 \approx U_L^2 \tau_t$, this damping rate is expressed as

$$1/\tau_{dam} \approx (\tau_t/\tau_a)^{\frac{1}{2}} (\Lambda/R) M/\tau_a. \tag{2.9a}$$

Another damping mechanism is due to the escape of energy from the vessel by sound radiation into the external medium from the open end of the tube, the order of magnitude of the corresponding damping rate is (see Rayleigh 1945):

$$1/\tau_{\mathrm{dam}} \approx (R/\Lambda)^2/\tau_{\mathrm{a}}. \tag{2.9b}$$

Notice that the dimensionless factors $(\Lambda/R)M$ and $(R/\Lambda)^2$ on the right-hand side of (2.9) are small numbers which may be of the same order of magnitude as βM on the right-hand side of (2.7). Taking into account that because of resonance phenomena, the instability is expected to be stronger for an acoustic time τ_{a} of the same order of magnitude as the transit time τ_{t} across the flame, the comparison between the growth rate (2.7) and the damping rates (2.9) shows that the instability may be produced in the following conditions:

$$\tau_{\mathrm{t}} \approx \tau_{\mathrm{a}} \Rightarrow 1/\beta \leqslant R/\Lambda \leqslant (\beta M)^{\frac{1}{2}}, \tag{2.10}$$

the first and second inequalities ensuring that the acoustic instability is stronger than the dissipation at the walls and the sound emitted from the open end respectively. According to (2.4*b*), the dimensionless numbers $1/\beta$ and $(\beta M)^{\frac{1}{2}}$ appearing in (2.10) are of a similar order of magnitude, 10^{-1}. Thus, according to this rough analysis based on orders of magnitude, an intrinsic instability of planar flame fronts may be expected to occur for sufficiently fast flames, $M \geqslant (1/\beta)^3 \approx 10^{-3}$, propagating in tubes with a typical aspect ratio. A definite answer may be provided only by a detailed analysis of this phenomenon, which is presented in the following sections of the paper.

3. Model and method

As we are interested in an intrinsic instability resulting from a coupling of the longitudinal acoustic modes with the internal structure of a planar front of a premixed flame propagating freely in a tube, the simplest model to consider is a one-dimensional reactive gaseous mixture. As is well known, premixed gas flames experience the Darrieus–Landau instability mechanism when wrinkled fronts are considered. The present analysis is limited to planar flame fronts which are relatively stable to those diffusive and hydrodynamical effects. The corresponding stability domains have been determined analytically by Pelcé & Clavin (1982), experimentally by Quinard, Searby & Boyer (1985) and numerically by Jackson & Kapila (1986).

The damping process associated with the wall effects are neglected in a first step. Viscous effects at the walls and sound emitted from the open end of the tube will be considered in §5. The two-dimensional edge effects associated with the penetration of the flame into the boundary layers described by Kaskan (1953) are also neglected here. Moreover, we consider the simplest chemical model of a flame based on a one-step exothermic and irreversible decomposition of a single reactant with kinetics governed by an Arrhenius law. The activation energy will be supposed much larger than the thermal energy. The starting point of the analysis is the conservation equations of mass, momentum and energy and the equation for the mass fraction of the reactant:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \tag{3.1a}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}+\frac{\partial\{(\eta+\eta')\,\partial u/\partial x\}}{\partial x}, \tag{3.1b}$$

$$\rho C_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=\frac{\partial(\lambda\,\partial T/\partial x)}{\partial x}+\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+(\eta+\eta')\left(\frac{\partial u}{\partial x}\right)^2+QW(T,\psi), \quad (3.1c)$$

$$\rho\left(\frac{\partial \psi}{\partial t}+u\frac{\partial \psi}{\partial x}\right)=\frac{\partial(\rho D\,\partial \psi/\partial x)}{\partial x}-W(T,\psi). \quad (3.1d)$$

Here η and η' are the first and second coefficients of dynamical viscosity, λ the heat conductivity, C_p the specific heat of the reactive mixture which is assumed to be constant for simplicity. D is the molecular diffusion coefficient and ψ the reduced mass fraction of the reactant which is assumed to be consumed by a one-step exothermic reaction ($\psi = 1$ in the fresh mixture, $\psi = 0$ in the burned gases). Q is the heat of reaction per unit mass of fresh mixture and W the reaction rate:

$$W = (\rho/\tau_c)\,\psi^n \exp(-E/\mathscr{R}T), \quad (3.1e)$$

n is the order of reaction, τ_c is proportional to the microscopic collision time. The reactive mixture is assumed to satisfy the ideal gas law:

$$p = (\mathscr{R}/\mathscr{M})\,\rho T \quad (3.1f)$$

where $\mathscr{R}$ is the constant of perfect gas and $\mathscr{M}$ the molecular weight of the reactive mixture which is assumed to be constant for simplicity.

When compressibility effects associated with pressure variations in the energy equation (3.1*c*) are neglected, the system of equations (3.1) may be reduced to a system of the 'reaction–diffusion' type. For this case, the travelling wave solution was first obtained by Zel'dovich & Frank-Kamenetskii (1938) for a large value of the dimensionless activation energy, $E/\mathscr{R}T_2 \gg 1$. The large parameter to be used is more precisely the Zel'dovich number defined as $\beta = E(T_2-T_1)/\mathscr{R}T_2^2$, where T_2 and T_1 are the temperature in the burned and fresh gaseous mixture. In the asymptotic limit $\beta\rightarrow\infty$, the flame structure is made up of two different lengthscales: an upstream non-reactive preheated zone of thickness d followed by a thin reaction–diffusion region of thickness d/β. The analysis of the former region is easy owing to its linear character. The analysis of the inner reaction–diffusion layer yields the heat flux produced by the exothermic reaction. Then, the final result is obtained by matching the two regions to give for a reaction of order unity:

$$\tau_t = (\beta^2/2Le)\,\tau_r \quad \text{and} \quad U_L = (\rho_2/\rho_1)\,(D_2/\tau_t)^{\frac{1}{2}}, \quad d = (\rho_2/\rho_1)\,(D_2\,\tau_t)^{\frac{1}{2}}, \quad (3.2)$$

where Le is the Lewis number defined as the ratio of thermal and molecular diffusivities, $Le = \lambda/\rho C_p D$. The isobaric approximation appears to be well justified because, according to the order of magnitude of the sound speed in the burned mixture, $c_2 \approx (D_2/\tau_c)^{\frac{1}{2}}$, the Mach number, U_L/c_2, computed from (3.2) and (2.3), is a small number for ordinary values of the activation energy. More precisely, let $M \equiv U_L/c_1$ be the Mach number defined in the fresh mixture, then (3.2) yields

$$M \approx \beta^{-1}\exp(-E/2\mathscr{R}T_2). \quad (3.3)$$

In order to describe the phenomena leading to the strong instability (2.8) through the cumulative mechanism presented in the previous section, we proceed as follows. Equations (3.1) are first reduced by using the characteristic length and time, d and τ_t, of the basic steady travelling wave solution: the isobaric planar flame. Then, the time-dependent solution of the non-dimensionalized version of (3.1) is considered in the asymptotic limit $\beta\rightarrow\infty$ but with the small number βM considered mathematically as of order unity even if this quantity is transcendentally small according to (3.3). Moreover, the driving mechanism of the vibratory instability having an intensity proportional to the gas expansion coefficient which is close to unity for ordinary flame conditions, $(\rho_1-\rho_2)/\rho_1 \approx (T_2-T_1)/T_2 \approx 0.8$, this quantity will be considered as a non-zero coefficient in the present analysis:

$$\epsilon \equiv \beta M = O(1) \quad \text{and} \quad (\rho_1-\rho_2)/\rho_1 = O(1), \quad \text{in the limit} \quad \beta \to \infty. \tag{3.4a}$$

As one is concerned with acoustic times never much shorter than the transit time, the reduced transient terms are here considered as quantities of order unity ($\tau_t/\tau_a = O(1)$). Moreover, acoustic phenomena with wavelengths Λ involve lengthscales which are much larger than the flame thickness, $\Lambda/d \equiv (\Lambda/c)(U_L/d)(c/U_L) = O(M^{-1})$,

$$\Lambda/d = (\tau_a/\tau_t)M^{-1} \quad \text{with} \quad (\tau_a/\tau_t) = O(1). \tag{3.4b}$$

In the limit $\beta \to \infty$, the reaction is still concentrated in a thin quasi-steady reaction–diffusion region of thickness d/β which may be properly treated as a discontinuity. Then, a similar method to the one introduced by Zel'dovich and Frank-Kamenestkii (1938) can be used to solve (3.1) in the limit (3.4*a*, *b*).

In order to keep gas expansion effects and to get rid of the variable coefficient in the time-dependent term, the following dimensionless mass-weighted coordinate is introduced as in the work of McIntosh (1986):

$$\xi = \frac{1}{\rho_1 d}\int_\alpha^x \rho(x',t)\,\mathrm{d}x', \tag{3.5a}$$

where $\alpha(t)$ is the position of the reaction–diffusion sheet:

$$\left.\begin{aligned} \frac{\partial}{\partial x} &\to d^{-1}\imath\frac{\partial}{\partial \xi}, \\ \frac{\partial}{\partial t}+u\frac{\partial}{\partial x} &\to \tau_t^{-1}\frac{\partial}{\partial \tau}+\tau_t^{-1}m(\tau)\frac{\partial}{\partial \xi}, \end{aligned}\right\} \tag{3.5b}$$

where τ is the reduced time $\tau = t/\tau_t$, $\imath(\xi,\tau) \equiv \rho(x,t)/\rho_1$ the reduced density and $m(\tau)$ the reduced mass flux at the reaction sheet: $m(\tau) \equiv \rho(x=\alpha,t)\{u(x=\alpha,t)-\mathrm{d}\alpha/\mathrm{d}t\}/\rho_1 U_L$. Outside the reaction–diffusion region, the conservation equations (3.1) are written in the dimensionless form as

$$\frac{\partial \imath^{-1}}{\partial \tau}+m\frac{\partial \imath^{-1}}{\partial \xi} = \frac{\partial u}{\partial \xi}, \tag{3.6a}$$

$$\frac{\partial u}{\partial \tau}+m\frac{\partial u}{\partial \xi} = -M^{-1}\frac{\partial \pi}{\partial \xi}+Pr\frac{\partial^2 u}{\partial \xi^2}, \tag{3.6b}$$

$$\frac{\partial \theta}{\partial \tau}+m\frac{\partial \theta}{\partial \xi}-\frac{\partial^2 \theta}{\partial \xi^2} = aM\left\{\imath^{-1}\left(\frac{\partial \pi}{\partial \tau}+m\frac{\partial \pi}{\partial \xi}\right)+MPr\left(\frac{\partial u}{\partial \xi}\right)^2\right\}, \tag{3.6c}$$

$$\frac{\partial \psi}{\partial \tau}+m\frac{\partial \psi}{\partial \xi}-(1/Le)\frac{\partial^2 \psi}{\partial \xi^2} = 0, \tag{3.6d}$$

with $Le = \lambda/\rho C_p D$ and with the following boundary conditions:

fresh gases, $\quad \xi \to -\infty: \quad \psi = 1, \quad \theta = O(M), \qquad (3.6e)$

burned gases, $\quad \xi \to +\infty: \quad \psi = 0, \quad \theta = 1+O(M). \qquad (3.6f)$

Here, θ is the reduced temperature $\theta \equiv (T-T_1)/(T_2-T_1)$ with $T_2-T_1 \equiv Q/C_p$, $Pr \equiv$

$(\eta+\eta')C_p/\lambda$ is the Prandtl number, $\mathscr{u} \equiv u/U_L$ is the reduced flow velocity, and $\pi \equiv p/\rho_1 c_1 U_L$ is a non-dimensional pressure obtained by scaling the pressure by the acoustic pressure corresponding to an amplitude $\delta u = U_L$. As the fresh mixture is considered as a perfect gas, the sound speed is given by $c_1^2 \equiv (C_p/C_v)(\mathscr{R}/\mathscr{M})T_1$ and the coefficient a appearing in the right-hand side of (3.6*c*), is defined as

$$a \equiv [((C_p-C_v)/C_v)(T_1/(T_2-T_1))]. \tag{3.6g}$$

All the diffusion coeffients D in (3.6*c*, *d*) have been assumed to verify $\rho^2 D$ = constant for simplicity. Another law could have been used but would not have changed the essential features of the final results (Nicoli & Pelcé 1989). The reduced form of the ideal gas law (3.1*f*) is

$$1+M(C_p/C_v)\,\delta\pi = r\{1+((T_2-T_1)/T_1)\,\theta\}, \tag{3.6h}$$

with by definition $\delta\pi \equiv \pi-\pi_1$ where $\pi_1 = p_1/\rho_1 c_1 U_L$ and $p_1 \equiv (\mathscr{R}/\mathscr{M})\rho_1 T_1$ is the initial pressure of the fresh mixture.

At the reaction sheet, the ordinary jump relations for the reduced temperature θ and mass fraction ψ are statisfied:

$$\left[\frac{\partial\theta}{\partial\xi}+\left(\frac{1}{Le}\right)\frac{\partial\psi}{\partial\xi}\right]_{\xi=0+}-\left[\frac{\partial\theta}{\partial\xi}+\left(\frac{1}{Le}\right)\frac{\partial\psi}{\partial\xi}\right]_{\xi=0-}=0, \tag{3.7a}$$

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0-}-\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0+}=\exp\{-\tfrac{1}{2}\beta(1-\theta_f)\}, \tag{3.7b}$$

$$\theta_f \equiv \theta|_{\xi=0+} = \theta|_{\xi=0-}, \quad \psi|_{\xi=0-} = \psi|_{\xi=0-} = 0, \tag{3.7c}$$

pressure and velocity are continuous at the reaction sheet located at $\xi = 0$.

As is well known, the analysis requires $\beta(Le-1) = O(1)$ (see Joulin & Clavin 1979 for example). The downstream and upstream sides of the reaction sheet are denoted by $\xi = 0+$ and $\xi = 0-$ respectively. Equation (3.7*a*) expressing the total enthalpy conservation across the reaction region, is valid at the two first leading orders in the asymptotic analysis. In (3.7*b*) expressing the kinetic law governing the heat flux released by the exothermic reaction, we have only retained the flame temperature modification ($\theta_f \neq 1$). The pressure variation of the pre-exponential factor of the reaction rate is neglected because its effect only appears at the next order.

It is sufficient for the following to determine the stationary solution of the freely propagating flame at the zeroth order in an expansion at small Mach number. As explained in §2, the relative amplitude of the perturbations of the flame structure due to the incoming acoustic wave are proportional to βM, thus much larger than the first correction to the stationary solution due to compressibility effects which are of order of magnitude βM^2. This is clearly exhibited by the reduced form (3.6*b*) of the momentum conservation equation: in the acoustic region one has according to (3.4*b*), $\xi = O(M^{-1})$, $\partial/\partial\xi = O(M)$ in such a way that $\delta\pi = O(1)$, but in the preheated region of the flame one has $\xi = O(1)$, $\partial/\partial\xi = O(1)$ and thus $\delta\pi = O(M)$ (for details see Nicoli & Pelcé 1989). Thus, one can neglect the pressure gradient inside the flame as well as the dissipation by viscous friction in the energy equation. The profiles of temperature and concentration of the steady travelling wave solution are obtained at leading order by solving (3.6*c*, *d*) with $M = 0$ and using the boundary conditions (3.6*e*, *f*) and the jump conditions (3.7), to give

$$\xi < 0: \quad \bar{\theta} = \exp\xi; \quad \xi > 0: \quad \bar{\theta} = 1, \tag{3.8a}$$

$$\xi < 0: \quad \bar{\psi} = 1-\exp(Le\,\xi); \quad \xi > 0: \quad \bar{\psi} = 0; \tag{3.8b}$$

the density and velocity profiles are given by (3.6*h*) and (3.6*a*):

$$\bar{r}(\xi)\left\{1+\frac{T_2-T_1}{T_1}\bar{\theta}(\xi)\right\}=1, \quad \bar{r}(\xi)\,\bar{u}(\xi)=1. \tag{3.8c}$$

The expression of the reduced mass flux, $\bar{m}=1$, results from the choice of the non-dimensionalization by length- and timescales (3.2) of the unperturbed planar flame.

4. The transfer function

4.1. *Scaling and regions*

When a sound wave of wavelength Λ and frequency ω ($\Lambda \approx L$ and $\omega \approx 1/\tau_a$ with $\tau_a \equiv L/c_1$, where L is the length of the tube) interacts with a premixed gas flame, the profiles of physical quantities are perturbed inside the flame and the flow velocity varies on two different lengthscales. Such a scale separation at small Mach number, which was mentioned at the end of the previous section, has already been largely discussed by Nicoli & Pelcé (1989) in the case of the Rijke tube where the flame is replaced by a hot grid. This question is discussed here only briefly. For $\omega\tau_t$ of order unity (which is the case in experiments), and according to (3.4*b*), the flame thickness d becomes very small compared to Λ when the Mach number is small. Two distinct regions appear.

The acoustic region on the lengthscale of the wavelength, Λ, $\xi = O(M^{-1})$, where perturbations are simply acoustic waves,

$$\delta\pi = O(1), \quad \delta u = O(1) \quad \text{but} \quad \delta\theta = O(M). \tag{4.1}$$

The flame region on the lengthscale d, $\xi = O(1)$, where the pressure has to be considered as uniform but time dependent and where

$$\delta\theta = O(1), \quad \delta u = O(1) \quad \text{but} \quad \delta\pi = O(M). \tag{4.2}$$

In the acoustic regions, the perturbations correspond to acoustic waves whose pressure and velocity fields may be easily computed in the linear approximation (see §5.1). At the scale of the acoustic wavelength, the flame appears as a hydrodynamical discontinuity for the acoustic flow velocity as well as for the gas density and temperature. A full determination of the acoustic field in the tube necessitates knowing the relation linking the pressure fluctuation at the flame front with the jump across the flame of the acoustic flow velocity. Such a transfer function is provided by an asymptotic analysis of the perturbed flame structure in the limit (3.4*a*, *b*).

4.2. *The transfer function*

The analysis starts with the linear decomposition of the reduced temperature and concentration profiles: $\theta = \bar{\theta}(\xi)+\delta\theta$, $\psi = \bar{\psi}(\xi)+\delta\psi$, where the unperturbed solutions $\bar{\theta}$ and $\bar{\psi}$ are given by (3.8*a*, *b*) and where the perturbations are decomposed in Fourier modes, $\delta\theta = \theta'(\xi)\exp(\mathrm{i}\omega\tau)$, $\delta\psi = \psi'(\xi)\exp(\mathrm{i}\omega\tau)$. Here ω is the reduced acoustic frequency, $\omega = \omega\tau_t$. In a similar way one introduces $\delta\pi = \pi'\exp(\mathrm{i}\omega\tau)$ for the reduced perturbation of the acoustic pressure at the flame and $m = 1+m'\exp(\mathrm{i}\omega\tau)$ for the modified reduced combustion rate. As usual, the leading order of the modification of the flame structure is given by the first order $1/\beta$ in the asymptotic expansion of the reduced temperature perturbation $\delta\theta$. Thus, according to (3.4*a*), the reduced linearized equations governing the perturbation of the flame structure are obtained from (3.6*c*, *d*) in the following form which is valid up to order $1/\beta$:

$$\mathrm{i}\omega\theta' + \frac{\mathrm{d}\theta'}{\mathrm{d}\xi} - \frac{\mathrm{d}^2\theta'}{\mathrm{d}\xi^2} = -m'\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\xi} + \frac{aM}{\bar{r}}\mathrm{i}\omega\pi', \tag{4.3a}$$

$$\mathrm{i}\omega\psi' + \frac{\mathrm{d}\psi'}{\mathrm{d}\xi} - \frac{1}{Le}\frac{d^2\psi'}{\mathrm{d}\xi^2} = -m'\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\xi}. \tag{4.3b}$$

At the reaction zone, $\xi = 0$, the linearized boundary conditions (3.7) apply in the form

$$\left[\frac{\mathrm{d}\theta'}{\mathrm{d}\xi} + \frac{1}{Le}\frac{\mathrm{d}\psi'}{\mathrm{d}\xi}\right]_{\xi=0_-}^{\xi=0_+} = 0, \tag{4.4a}$$

$$\left.\frac{\mathrm{d}\theta'}{\mathrm{d}\xi}\right|_{\xi=0_-} = \tfrac{1}{2}\beta\theta'(\xi=0). \tag{4.4b}$$

Temperature, mass fraction of the reactant, and flow velocity disturbances are continuous at the reaction sheet at $\xi = 0$. Integration of (4.3*a*) and (4.3*b*) leads to

$$\xi < 0: \quad \theta' = A\exp(\chi_+(1)\,\xi) - \frac{m'}{\mathrm{i}\omega}\exp\xi + \frac{aM}{\bar{r}}\pi', \tag{4.5a}$$

$$\xi > 0: \quad \theta' = B\exp(\chi_-(1)\,\xi) + (T_2/T_1)\,aM\pi', \tag{4.5b}$$

$$\xi < 0: \quad \psi' = C\exp(\chi_+(Le)\,\xi) + \frac{m'}{\mathrm{i}\omega}Le\exp(Le\,\xi); \quad \xi > 0: \quad \psi' = 0, \tag{4.5c}$$

where
$$\chi_\pm(Le) = \tfrac{1}{2}Le\left[1 \pm \left(1 + \frac{4\mathrm{i}\omega}{Le}\right)^{\frac{1}{2}}\right]. \tag{4.6}$$

The temperature and concentration profiles (4.5*a*, *b*, *c*) match the upstream ($\xi \to -\infty$) and downstream ($\xi \to +\infty$) non-reactive acoustic zones where the temperature fluctuations correspond to the isentropic compression by acoustic waves. Coefficients A, B can be expressed in terms of $\theta'_{\mathrm{f}}(\xi = 0)$. Coefficient C is determined by the boundary condition (3.7*c*), $\psi'(\xi = 0) = 0$. Then, $\theta'_{\mathrm{f}}(\xi = 0)$ and the perturbed mass flow rate m' are determined by the two jump conditions (4.4) at the reaction sheet to give

$$m' = \beta M\left(\frac{T_2}{T_2 - T_1}\right)\left(\frac{C_p - C_v}{C_v}\right) \times \left[\frac{\{(1+4\mathrm{i}\omega)^{\frac{1}{2}} - (\rho_1 - \rho_2)/\rho_1\}\,(1+4\mathrm{i}\omega)^{\frac{1}{2}}}{[(1+4\mathrm{i}\omega)^{\frac{1}{2}} - 1]\,(1+4\mathrm{i}\omega) - \frac{1}{2}\beta(Le-1)\,[1 - (1+4\mathrm{i}\omega)^{\frac{1}{2}} + 2\mathrm{i}\omega]}\right]\mathrm{i}\omega\pi', \tag{4.7}$$

where according to (3.4*a*) βM is considered as a quantity of order unity in the limit $\beta \to \infty$.

This result generalizes the one previously obtained by Van Harten *et al.* (1984, see their equation (67)) which corresponds to (4.7) in the limit of a small gas expansion, $(\rho_1 - \rho_2)/\rho_1 = (T_2 - T_1)/T_2 \to 0$ but $M\beta T_2/(T_2 - T_1) = O(1)$.

The denominator of (4.7) corresponds to the dispersion relation of the planar flame stability. A resonance phenomenon appears at $\omega = \omega_{\mathrm{c}}$ when $\beta(Le-1)$ approaches from below the critical value $\beta(Le_c - 1) \equiv 4(1+\sqrt{3})$ corresponding to a Hopf bifurcation at which the planar flame becomes unstable under diffusive–thermal effects with a natural oscillatory frequency $\omega_c = O(1)$. Such an instability was pointed out by Sivashinsky (1977) and Joulin & Clavin (1979) in the framework of the diffusive–thermal model $(\rho_1 - \rho_2)/\rho_1 = 0$. A by-product of the present analysis is to show that this instability is not modified by the gas expansion. But, in fact, because of the relatively high value of the corresponding critical Lewis number, Le_c, such a resonance phenomenon is not easily accessible for ordinary gaseous reactive mixtures. The present analysis is limited to cases where $Le < Le_c$.

The pressure-induced fluctuations of the heat flux are

$$\left.\frac{\mathrm{d}\theta'}{\mathrm{d}\xi}\right|_{\xi=0-} = [(\chi_+(1)-1)/\mathrm{i}\omega]\, m'. \tag{4.8}$$

The leading order of the acoustic velocity jump across the flame is obtained by a space integration (between $\xi = -\infty$ and $\xi = 0_-$) of the linearized continuity equation (3.6*a*) which, by using the ideal gas law (3.6*h*), can be written as

$$\left(\frac{T_2 - T_1}{T_1}\right)\left(\mathrm{i}\omega\theta' + \frac{\mathrm{d}\theta'}{\mathrm{d}\xi} + m'\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\xi}\right) = \frac{\mathrm{d}u'}{\mathrm{d}\xi}. \tag{4.9a}$$

The result of this integration can be directly obtained from the leading order in the asymptotic expansion (4.3*a*) in the form

$$u_2' - u_1' = \left(\frac{T_2 - T_1}{T_1}\right)\left.\frac{\mathrm{d}\theta'}{\mathrm{d}\xi}\right|_{\xi=0_-} \tag{4.9b}$$

and (4.8) and (4.7) lead to the final result which can be written in dimensional form as

$$\delta u_2 - \delta u_1 = Z(\omega\tau_t)\frac{\delta p}{\rho_1 c_1} \tag{4.10a}$$

where the transfer function $Z(\omega\tau_t)$ is a complex scalar which is given by

$$Z(\omega\tau_t) = \tfrac{1}{2}\beta M \frac{T_2}{T_1}\left(\frac{C_p - C_v}{C_v}\right) \times \left[\frac{[(1+4\mathrm{i}\omega\tau_t)^{\frac{1}{2}} - 1]\{(1+4\mathrm{i}\omega\tau_t)^{\frac{1}{2}} - (\rho_1-\rho_2)/\rho_1\}(1+4\mathrm{i}\omega\tau_t)^{\frac{1}{2}}}{[(1+4\mathrm{i}\omega\tau_t)^{\frac{1}{2}} - 1](1+4\mathrm{i}\omega\tau_t) - \frac{1}{2}\beta(Le-1)[1-(1+4\mathrm{i}\omega\tau_t)^{\frac{1}{2}} + 2\mathrm{i}\omega\tau_t]}\right]. \tag{4.10b}$$

The relation (4.9*b*) shows that, as anticipated at the beginning of §2, the velocity jump is proportional to the heat flux with a real and positive coefficient. Thus (4.7) and (4.8) can be directly used to test the Rayleigh criterion. Instability occurs when heat flux and pressure are in phase, i.e. if the real part of the transfer function Z is positive. Notice that in the limit of a small gas expansion used by Van Harten *et al.* (1984) (see the text under (4.7)), the transfer function goes to zero and the driving mechanism of the vibratory instability disappears.

The limiting cases of low and high frequencies of the transfer function correspond to $Z = (\frac{1}{2}\beta M)(C_p - C_v)/C_v$ and $Z = (\frac{1}{2}\beta M)(T_2/T_1)(C_p - C_v)/C_v$ respectively. The first case corresponds to a quasi-stationary modification of the combustion rate by the temperature modification produced by the acoustic wave. The real and imaginary parts of $2Z/\beta M$ are plotted in figure 1(*a*) for different values of the Lewis number. The driving mechanism of the oscillatory instability appears to become maximum in the region where the acoustic frequency approaches the inverse of the transit time, $\omega\tau_t = O(1)$. Moreover, because of the thermal–diffusive resonance phenomenon, the maximum intensity of the vibratory instability is found to be amplified for sufficiently large Lewis numbers, which correspond to reactive mixtures with a limiting species much heavier than the diluent as, for example, in lean mixtures of heavy hydrocarbons in air.

5. Vibratory flame instability

5.1. *Eigenmode equation in the non-dissipative case*

In a first step, the mean position of the planar flame front is assumed to be kept at rest in the laboratory frame by adjusting the mean flow velocity of the incoming fresh gases at the laminar flame speed U_L. This section is devoted to the stability analysis of such a planar flame in the presence of axial acoustic modes of the tube and the approximation where viscous effect are neglected. The analysis requires the solution of (3.6) in the acoustic regions. These external solutions correspond to the acoustic field described in dimensionalized form by the linearized equations (3.1*a*, *b*) in the isentropic approximation to give†

$$\delta p_{1,2} = \left\{A_{1,2}\exp\left(\mathrm{i}\frac{\omega}{c_{1,2}}x\right)+B_{1,2}\exp\left(-\mathrm{i}\frac{\omega}{c_{1,2}}x\right)\right\}\exp \mathrm{i}\omega t, \tag{5.1a}$$

$$\delta u_{1,2} = -\frac{1}{\rho_{1,2}c_{1,2}}\left\{A_{1,2}\exp\left(\mathrm{i}\frac{\omega}{c_{1,2}}x\right)-B_{1,2}\exp\left(-\mathrm{i}\frac{\omega}{c_{1,2}}x\right)\right\}\exp \mathrm{i}\omega t, \tag{5.1b}$$

where subscripts 1 and 2 denote the unburned mixture and the burned gases respectively. As usual, the constants of integration $A_{1,2}$ and $B_{1,2}$ have to be determined by the boundary conditions. As already discussed, the flame including its preheated zone appears, at the leading order of the external acoustic solutions, as a discontinuity separating the fresh mixture from the burned gases. The boundary conditions at the mean flame position, chosen at the origin of the coordinate ($x = 0$), result from the inner structure analysis of the previous section. Pressure variations can be considered as equal on both sides of the discontinuity and the velocity jump is given by (4.10*a*):

$$\delta p_1(0) = \delta p_2(0) \equiv \delta p(0), \tag{5.2}$$

$$\delta u_2(0)-\delta u_1(0) = Z(\omega\tau_t)\,\delta p(0)/\rho_1 c_1, \tag{5.3}$$

where the transfer function $Z(\omega\tau_t)$ is given by (4.10*b*). Other boundary conditions hold at the ends of the tube. For simplicity, we consider in this paper only the case of a flame propagating from the open to the closed end of a tube which corresponds to the classical experimental situation (see for example Kaskan 1953). Other configurations can be investigated by the same method and will be presented elsewhere. In our case, the appropriate boundary conditions are

closed end in the fresh mixture,

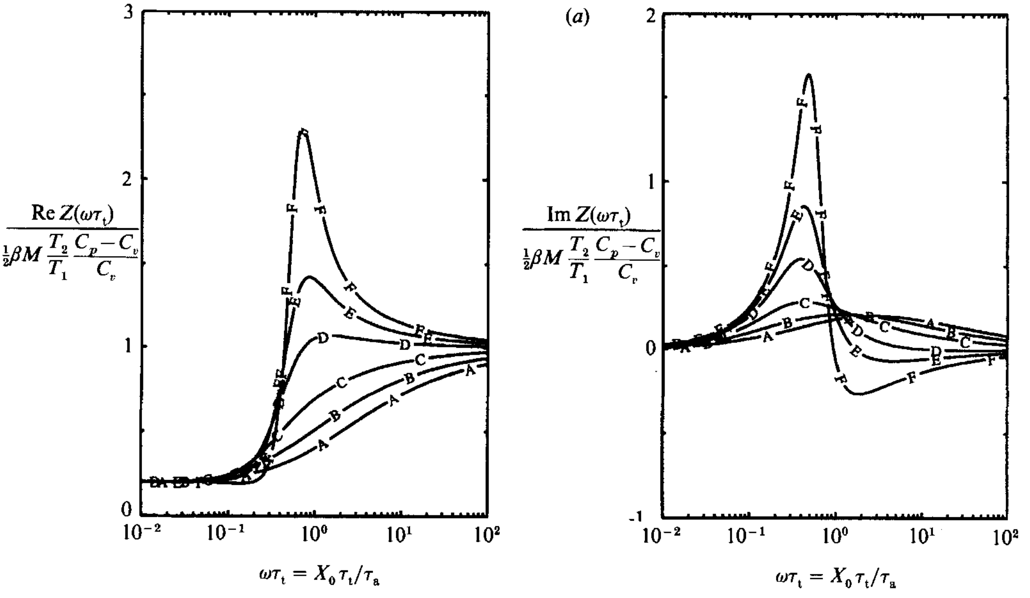
$$x = -rL: \quad \delta u_1 = 0, \tag{5.4a}$$

open end in the burnt gases,

$$x = (1-r)L: \quad \delta p_2 = 0. \tag{5.4b}$$

† The effects of non-zero mean flow velocity are negligible because they induce relative frequency and phase shifts of order M^2 and M respectively.

(a)



(b)

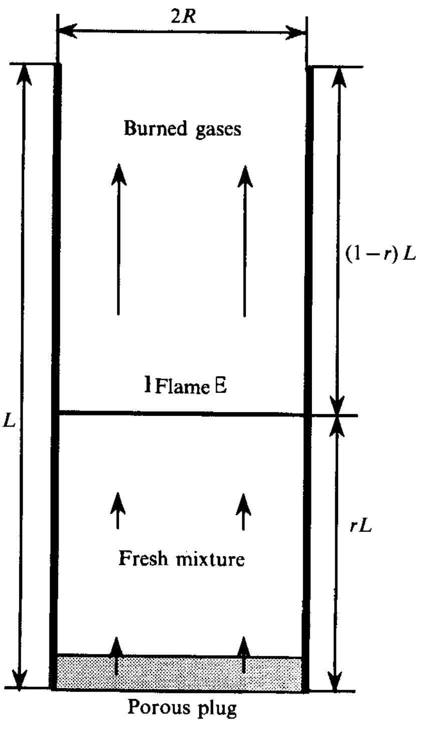
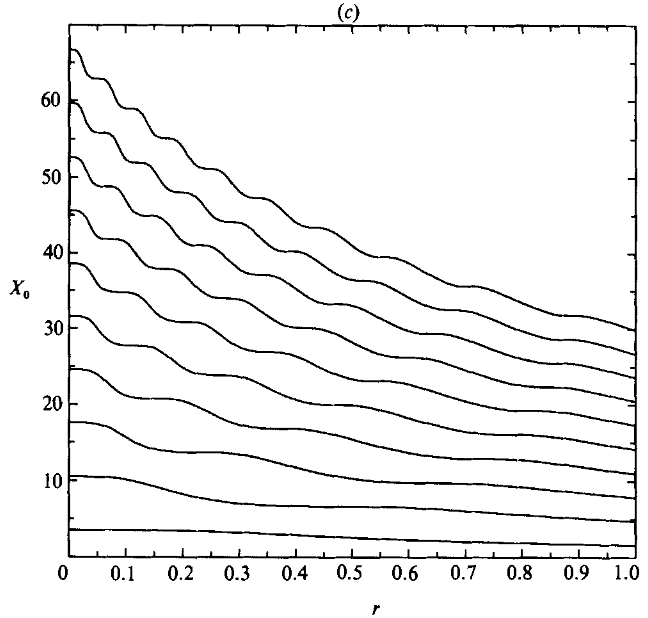


FIGURE 1 (a, b). For caption see facing page.



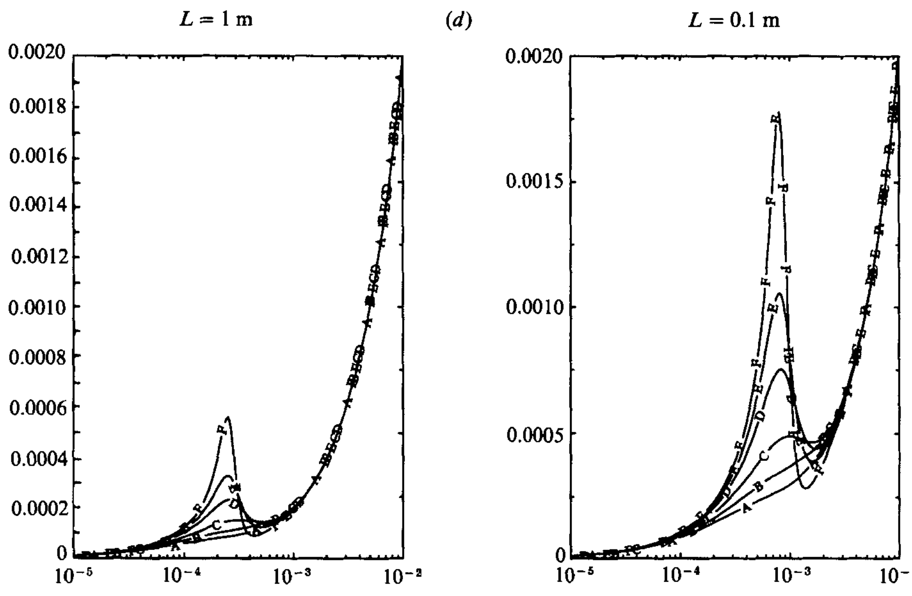


FIGURE 1. (*a*) Variation with frequency of real and imaginary parts of the reduced transfer function (4.10*b*) for different values of Lewis number and for $\beta = 20$: A, $Le = 0.6$; B, 0.8; C, 1; D, 1.2; E, 1.3; F, 1.4. (*b*) Flame configuration. (*c*) Eigenmode $X_0(r)$ solution of (5.6). (*d*) Variation with Mach number of the non-dimensional growth rate

$$\mathrm{Re}\,(Z)/\tfrac{1}{2}\beta\frac{T_2}{T_1}\frac{C_p-C_v}{C_v}$$

for different values of Lewis number and for $\beta = 20$, $X_0 = 2$: A, $Le = 0.6$; B, 0.8; C, 1; D, 1.2; E, 1.3; F, 1.4.

Such conditions would correspond to the case shown in figure 1 (*b*) where the fresh mixture flows through a porous plug standing far away from the flame position ($rL \gg d$).

When the constants $A_{1,2}$ and $B_{1,2}$ are eliminated by using the boundary conditions (5.2), (5.3) and (5.4), one obtains the following eigenmode equation:

$$\frac{\rho_2 c_2}{\rho_1 c_1}\tan(rX)\tan\left((1-r)\frac{c_1}{c_2}X\right) = 1 - \mathrm{i}\frac{\rho_2 c_2}{\rho_1 c_1}Z(X\tau_t/\tau_a)\tan\left((1-r)\frac{c_1}{c_2}X\right). \tag{5.5}$$

Here X is the dimensionless frequency $X = \omega\tau_a$, $\tau_a \equiv L/c_1$ the characteristic acoustic time. By definition, X is of unity but according to (4.10*b*), the dimensionless transfer function Z is proportional to $\epsilon \equiv \beta M$ whose the characteristic order of magnitude is 10^{-2} (see (2.4*b*)). Thus the eigenmodes can be computed from (5.5) by an expansion† in ϵ around the free eigenmode X_0 solutions of

$$\frac{\rho_2 c_2}{\rho_1 c_1}\tan(rX_0)\tan\left((1-r)\frac{c_1}{c_2}X_0\right) = 1. \tag{5.6}$$

X_0 are real numbers which characterize the acoustic frequencies of the tube when the flame is considered as a passive interface separating two different gaseous media. The corresponding eigenmodes are plotted in figure 1 (*c*) for the first ten harmonics. Writing $X = X_0 + \delta X$, one obtains at the first order in the power expansion of ϵ:

$$\mathrm{Im}\,(\delta X) = -\frac{\mathrm{Re}\,(Z)}{r(1+\tan^2 rX_0) + \frac{\rho_2 c_2}{\rho_1 c_1}(1-r)\frac{c_1}{c_2}\left(1+\tan^2(1-r)\frac{c_1}{c_2}X_0\right)\tan^2 rX_0}. \tag{5.7}$$

Instability corresponds to $\mathrm{Im}\,(\omega) < 0$, i.e. $\mathrm{Im}\,(\delta X) < 0$. According to (5.7), the instability is found to appear when $\mathrm{Re}\,(Z) > 0$, which is in agreement with the Rayleigh criterion as formulated at the end of §4. Moreover, (4.10*b*) shows that $\mathrm{Re}\,(Z)$ is effectively positive at all frequencies (see also figure 1). Thus, the coupling of the flame structure with the axial acoustic modes of the tube leads to a natural instability mechanism with oscillatory frequencies that are very close to those of the tube which are given by (5.6). The instability growth rate as given by (5.7) and (4.10*b*), is in agreement with the result (2.7) of the rough analysis presented in §2 and it turns out to be a small quantity compared to the natural acoustic frequency $1/\tau_a \equiv c_1/L$.

5.2. *Stability limits*

The instability can develop only when the linear growth rate (5.7) is sufficiently large to overcome damping effects. Transverse heat conduction and viscous friction at the tube walls are unavoidable dissipative phenomena which produce a natural damping of the instability. Another damping may be produced by a loss of acoustic energy because of sound radiation from the open end of the tube to the external medium. The dissipative effects at the tube walls involve transverse variations of the acoustic field which cannot be described by the one-dimensional model used in the previous part. In the absence of flame, the two-dimensional dissipation mechanisms of viscous friction and heat conduction to the tube walls were computed a long time ago by Kirchhoff for axial acoustic waves in a cylindrical tube of circular section (see pp. 319 and 323 of Rayleigh 1945). When the tube radius is large compared to the thickness

† Notice that if βM had been scaled to be small from the beginning no additional term would be introduced in (3.6) and (3.7). Result (4.7) and (5.5) remain valid in the limit $\beta M \to 0$.

of the boundary layer $h \approx (\nu/\omega)^{\frac{1}{2}}$, the acoustic fields in the bulk of the tube take a similar form to (5.1), but with additional damping terms (see p. 326 of Rayleigh 1945):

$$\delta p_{1,2} = \{A_{1,2} \exp(n_{1,2}x) + B_{1,2} \exp(-n_{1,2}x)\} \exp \mathrm{i}\omega t, \tag{5.8a}$$

$$\delta u_{1,2} = -\frac{n_{1,2}}{\mathrm{i}\omega\rho_{1,2}} \{A_{1,2} \exp(n_{1,2}x) - B_{1,2} \exp(-n_{1,2}x)\} \exp \mathrm{i}\omega t, \tag{5.8b}$$

where
$$n_{1,2} = \mathrm{i}\frac{\omega}{c_{1,2}}(1+\phi_{1,2}), \quad \phi_{1,2} = \left\{Pr^{\frac{1}{2}} + \frac{(C_p - C_v)}{(C_p C_v)^{\frac{1}{2}}}\right\} \frac{D_{1,2}^{\frac{1}{2}}}{R(2\omega)^{\frac{1}{2}}}(1-\mathrm{i}). \tag{5.8c}$$

At the open end of the tube ($x = (1-r)L$), radiation of acoustic waves in the free atmosphere leads to the following boundary condition (Rayleigh 1945) which has to replace (5.4*b*):

$$\delta p_2 = K_2 \rho_2 c_2 \delta u_2 \quad \text{with} \quad K_2 = (\omega^2 R^2/2c_2^2 + 8\mathrm{i}\omega R/3\pi c_2), \tag{5.9}$$

where $\omega R/c_2$ is assumed a small number.

According to the order of magnitude estimates presented in (2.9) and (2.10), Re $(K_2) = O(R^2/L^2)$, Im $(K_2)^2 = O(R^2/L^2)$, Im $(\phi_2) = O((D_T \tau_a)^{\frac{1}{2}}/R)$ and $|Z| = O(\beta M)$ have to be considered at this stage of the analysis as small quantities of the same order of magnitude. The analysis of the stability limits may be carried out as in the previous section with the boundary conditions (5.2) and (5.3) at the flame front. An eigenmode equation follows, similar to (5.5) but where viscous and radiation terms are introduced as small perturbations. Thus eigenmodes may be computed by an expansion around the free eigenmode X_0 solutions of (5.6), to give

$$\begin{aligned} \mathrm{Im}(\delta X)&\left[r\{1+\tan^2(rX_0)\} + (1-r)\frac{\rho_2}{\rho_1}\left\{1+\tan^2\left(\frac{c_1}{c_2}(1-r)X_0\right)\right\}\tan^2(rX_0)\right] \\ &= -\mathrm{Re}(Z)(X_0\tau_t/\tau_a) - \mathrm{Im}(\phi_1)\{rX_0(1+\tan^2(rX_0)) + \tan(rX_0)\} \\ &\quad - \mathrm{Im}(\phi_2)\left\{(1-r)\frac{\rho_2}{\rho_1}X_0\left[\tan^2(rX_0) + \frac{\rho_1^2 c_1^2}{\rho_2^2 c_2^2}\right] - \tan X_0\right\} \\ &\quad + \mathrm{Re}(K_2)\frac{\rho_1 c_1}{\rho_2 c_2}\left\{1 + \left(\frac{\rho_2^2 c_2^2}{\rho_1^2 c_1^2}\right)\tan^2(rX_0)\right\}. \end{aligned} \tag{5.10a}$$

According to (5.8*c*) and (5.9), Im $(\phi_{1,2}) < 0$ and Re $(K_2) > 0$. Moreover, the coefficients $\{\}$ of Im $(\phi_{1,2})$ and of Re (K_2) in the above equation are positive. The second, third and fourth terms in the right-hand side of (5.10*a*) yield a positive contribution to Im (δX) corresponding to damping mechanisms associated with viscous friction, heat transfer at the walls and energy loss by sound radiation at the open end of the tube.

The stability limits of the flame in the tube are obtained from (5.10*a*) by setting Im $(\delta X) = 0$. In the presentation of the results, we have chosen to vary the following three parameters: the Mach number M, the Lewis number Le, and the aspect ratio R/L of the tube. Two values of the length of the tube are considered: $L = 1$ m and $L = 0.1$ m. The other parameters are kept fixed: $c_1 = 330$ m/s, $D_1 = 0.15\ 10^{-4}$ m²/s, $\beta = 20$, $T_2/T_1 = \rho_1/\rho_2 = 5$, $D_2/D_1 = 5^{\frac{3}{2}}$, $c_2/c_1 = \sqrt{5}$, $(C_p - C_v)/C_v = \frac{2}{3}$. All the quantities in (5.10*a*), such as Im $(\phi_{1,2})$ and Re (K_2) defined as (5.8*c*) and (5.9), have to be rewritten by using $\omega = X_0 c_1/L$,

$$-\mathrm{Im}(\phi_{1,2}) = (D_{1,2}/2c_1 L)^{\frac{1}{2}}(R/L)^{-1} X_0^{-\frac{1}{2}}, \tag{5.10b}$$

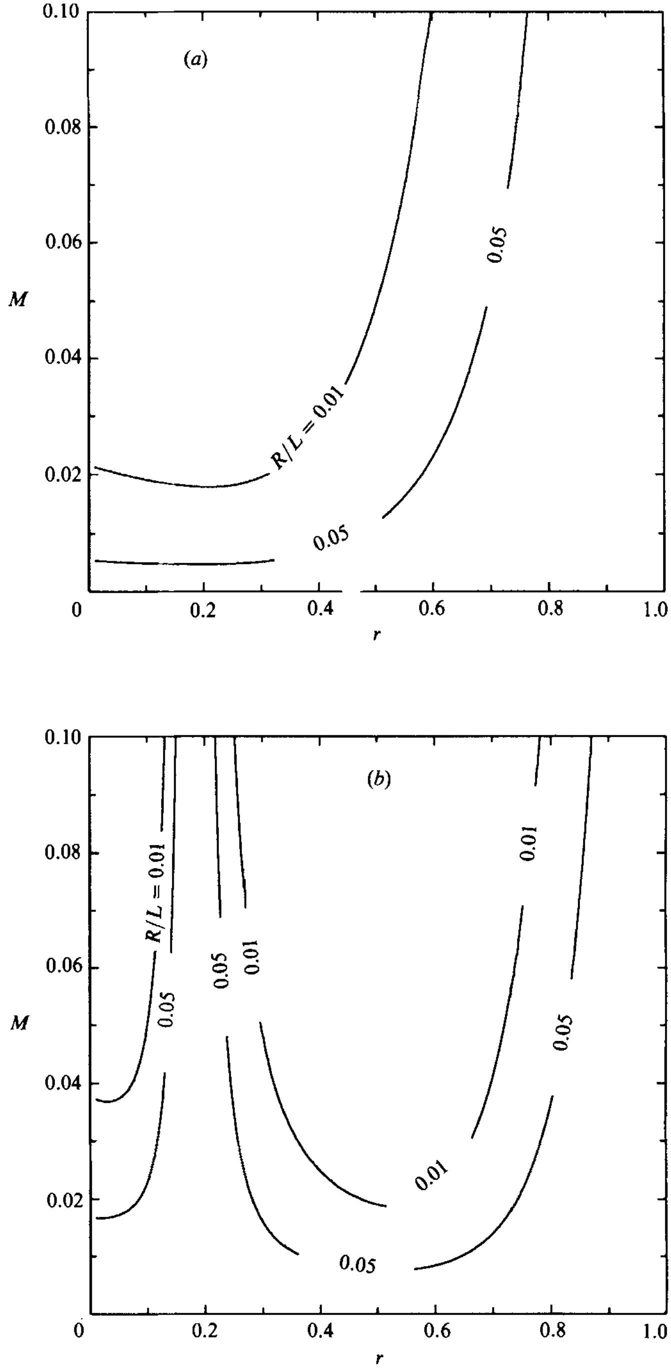


FIGURE 2. (*a*) Stability limits of the fundamental mode in the presence of loss by sound radiation. $L = 1$ m, $\beta = 20$, $Le = 1$, $R/L = 0.01$ and 0.05. (*b*) Stability limits of the second harmonic for the same conditions as in (*a*).

where the value of the bracket {} in (5.8*c*) has been taken equal to $\sqrt{2}$ for simplicity,

$$\mathrm{Re}\,(K_2) = 2.5(R/L)^2 X_0^2. \tag{5.10c}$$

In a similar way, using (3.2), the time ratio τ_t/τ_a in the argument of the transfer function appearing on the right-hand side of (5.10*a*), has to be expressed in the form

$$\left.\begin{aligned} \tau_t/\tau_a &= (D_2/c_1 L)(\rho_2/\rho_1)^2 M^{-2} \approx 2\times 10^{-8} L^{-1} M^{-2}, \\ \omega\tau_t &= X_0 \tau_t/\tau_a \approx 2\times 10^{-8} X_0 L^{-1} M^{-2} \end{aligned}\right\} \tag{5.10d}$$

in which L has to be expressed in metres. Then, one determines the stability limits for the plane Mach number M *vs.* relative flame position r, as follows: for a given value of r, one determines the first zero X_0 of (5.6) (see figure 1*c*) and then from (5.10) one determines the Mach number M for which $\mathrm{Im}\,(\delta X)$ vanishes. The instability appears when the reduced growth rate $\mathrm{Re}\,(Z)$ is sufficiently large to balance the damping terms whose the amplitude are not Mach-number dependent (see (5.10*b*, *c*)). The dependence on the Mach number of the reduced growth rate is plotted in figure 1 (*d*) for different values of the Lewis number and for two typical values of the length of the tube, $L = 1$ m and $L = 0.1$ m. In these figures, X_0 has been chosen fixed at a typical value $X_0 = 2$ of the first acoustic mode of the tube (see figure 1*c*). In the range of accessible Mach numbers for premixed flames, $M \leqslant 2\times 10^{-3}$, one can see a peak value of the growth rate which corresponds to the proximity of a resonance phenomenon with the thermal–diffusive instability of the flame structure corresponding to a critical Lewis number Le_c defined by $\beta(Le_c - 1) = 4(1+\sqrt{3}) \approx 10.93$ (see §4.2). As a result of the L-dependence of $\omega\tau_t$ (see (5.10*d*)), the corresponding peak is more pronounced for short tubes (see figure 1*d*).

The stability limits of the fundamental mode are plotted in figure 2 (*a*) for a tube length $L = 1$ m, a Lewis number $Le = 1$ and for different values of R/L. Instability is found to occur in the second half of the tube since the strength of the instability is maximum at an antinode of pressure, which is the case at the close end of the tube. Critical values of the Mach number being relatively high, of order 0.01, the corresponding values of τ_t/τ_a are, according to (5.10*d*), much smaller than unity. As a result, only the low-frequency limit of the transfer function is involved (see figure 1*a*). According to the results plotted in figure 1 (*d*), lower critical values of the Mach number and a stronger instability can be expected to occur in shorter tubes. The stability limits of the fundamental mode are plotted on figure 3 for a tube length ten times shorter, $L = 0.1$ m, and for two values of the Lewis number, $Le = 1$ (figure 3*a*) and $Le = 1.4$ (figure 3*b*) and when, in addition, energy loss by sound radiation has been suppressed. One may notice in this last case an island of strong instability at small Mach number ($M \approx 0.001$) corresponding to the maximum value of the real part of the transfer function. We shall come back to this phenomenon in the following section. For comparison, the stability limits of the first harmonic are plotted in figures 2 (*b*) and 3 (*b*).

5.3. *Flames propagating in tubes*

Consider now the case where the flame propagates at constant velocity from the opened to the closed end of the tube at small Mach number in a quiescent reactive mixture. For each relative position $r(t)$ of the flame, the imaginary part of the eigenmodes are determined adiabatically by relation (5.10*a*). Let the total growth of an initial perturbation be characterized by Γ defined as: $\exp\Gamma = A_f/A_i$, where A_f and A_i are the final and initial amplitude of the perturbation. Because of the large difference in timescales ($\beta M \approx 10^{-2}$) between the acoustic vibration frequency and

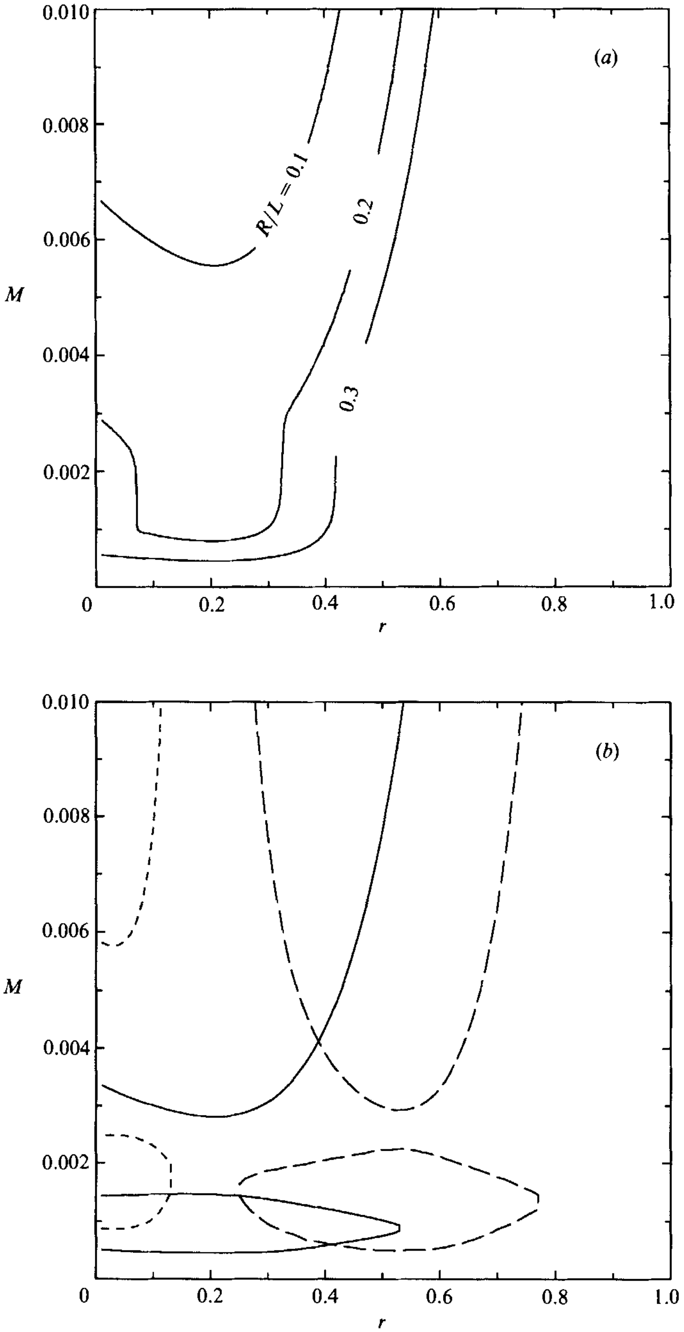


FIGURE 3. (*a*) Stability limits of the fundamental mode without loss by sound radiation ($K_2 = 0$). $L = 10$ cm, $\beta = 20$, $Le = 1$, $R/L = 0.1$, 0.2 and 0.3. (*b*) Stability limits without loss by sound radiation ($K_2 = 0$). $L = 10$ cm, $\beta = 20$, $Le = 1.4$, $R = 2$ cm. ——, fundamental mode; -----, second harmonic.

the growth rate of the vibratory instability, the total growth between the initial and final times t_i and t_f may be computed by an 'adiabatic approximation' in the form

$$\Gamma = -\int_{t_i}^{t_f} \mathrm{Im}\,(\omega)\,\mathrm{d}t \tag{5.11a}$$

or

$$\Gamma = \frac{1}{M}\int_{r_i}^{r_f} \mathrm{Im}\,(\delta X)\,\mathrm{d}r, \tag{5.11b}$$

where $r_{i,f} = -(U_L/L)\,t_{i,f}$.

Consider first the large-Mach-number limit of Γ, for which damping mechanisms in (5.10) are negligible, and where $\tau_t/\tau_a = (D_2/c_1 L)(\rho_2/\rho_1)^2 M^{-2}$ is close to zero, which corresponds to a quasi-static approximation for the flame structure. Then, according to (4.10*b*), (5.10*a*) and (5.11*b*), Γ is given by

$$\Gamma = \tfrac{1}{2}\beta\frac{(C_p - C_v)}{C_v}\int_{r_f}^{r_i} \frac{\mathrm{d}r}{r(1+\tan^2 rX_0) + \frac{\rho_2 c_2}{\rho_1 c_1}(1-r)\frac{c_1}{c_2}\left(1+\tan^2(1-r)\frac{c_1}{c_2}X_0\right)\tan^2 rX_0}. \tag{5.12}$$

This relation shows that Γ is a number of order β as mentioned in §2 (see (2.8)). This is the main result of the paper. The coupling between the acoustic field and the free flame propagation, through the effects on the burning velocity of the pressure-induced temperature variations, can lead to an observable instability. The order of magnitude of the coupling constant, βM, is small (typically 10^{-2}) but, when the corresponding small instantaneous growth rate is integrated on the long residence time ($t_{res} \approx \tau_a/M$) that the flame takes to propagate on a distance of the acoustic wavelength, the total growth of an initial perturbation may be very large in the limit $\beta \rightarrow \infty$.

But, from a quantitative point of view, by noticing that for $r_f = 0$ and $r_i = 0.5$ and 1, the value of the integral in (5.12) is 0.309 and 0.322 respectively, the numerical coefficient in front of β in the expression of Γ turns out to be of the order of 0.1. Thus, for large Mach numbers, the resulting numerical values of $\Gamma(\approx \beta/10)$ yields a total growth, $\exp \Gamma$, which is not actually large for typical values of the reduced activation energy $(10 \leqslant \beta \leqslant 20)$. In this case where the flame responds in a quasi-steady manner, the vibratory instability mechanism leads to a weak effect.

As mentioned in the previous section, a strong vibratory instability may appear at sufficiently small Mach number for which $\tau_t/\tau_a = O(1)$ and when the resonance phenomena described in figures 1 (*a*) and 1 (*d*) is involved in the unsteady response of the flame structure. In the simplified flame model used here, this is the case for Lewis numbers sufficiently large $\beta(Le-1) \geqslant 6$.

For example, the numerical values of Γ corresponding respectively to the instability domains of figures 2 (*a*) and 3 (*b*) are plotted from (5.11*b*) as a function of the Mach number in figures 4 (*a*) and 4 (*b*). The corresponding total amplification, exp Γ, is found to be almost negligible in the first case and can be very large in the second one in a relatively thin window of low Mach numbers. Thus, such strong vibratory instabilities appear to be very sensitive to the parameters, and are expected to occur only for particular reactive mixtures. Nevertheless, similar phenomena have already been reported in the numerical simulation (Kooker 1979) of ozone flames propagating in short tubes.

Let A_i be the initial amplitude of an initial acoustic disturbance which may be

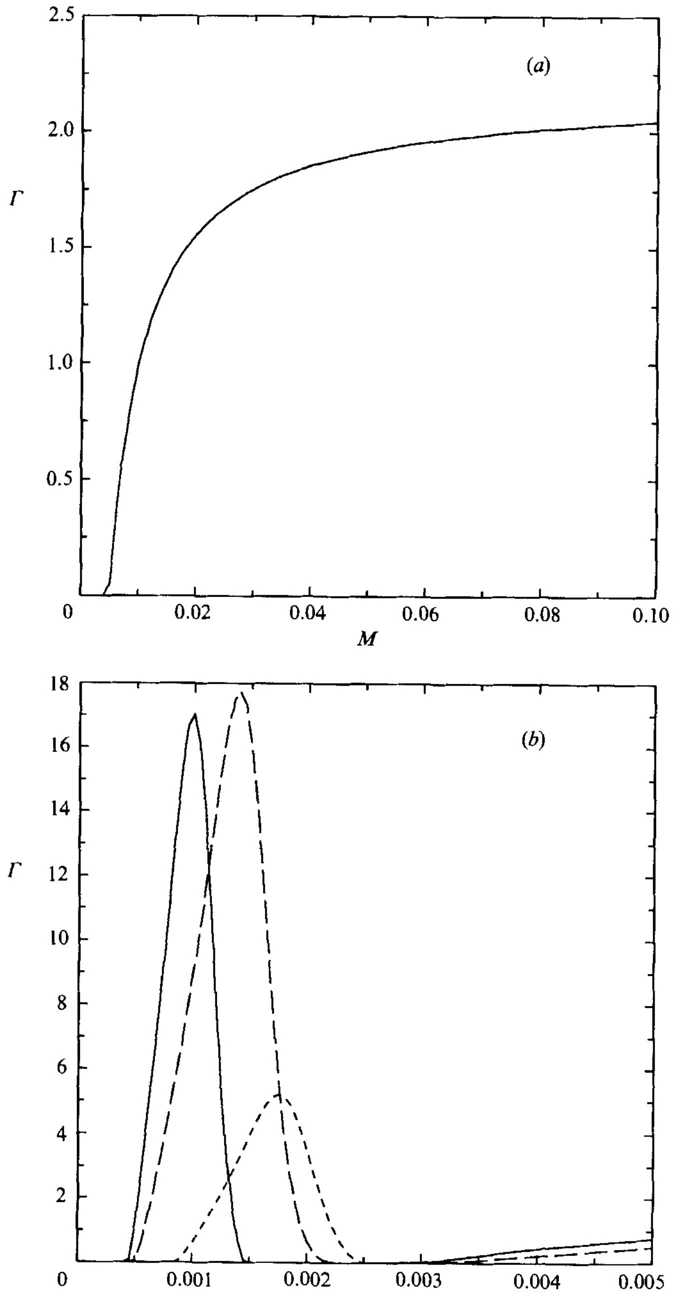


FIGURE 4. (*a*) Total growth rate for the same conditions as in figure 2(*a*) with $R/L = 0.05$. (*b*) Total growth rate for the same conditions as in figure 3(*b*). ——, fundamental mode; – – – –, second harmonic.

produced for example by the ignition process. The criterion defining the instability threshold is given by a final amplitude of the disturbance reaching a prescribed observable value of order unity in an appropriate system of unity. As $A_f \approx A_i \exp \Gamma$, it follows that the instability threshold corresponds to $\Gamma \approx -\log A_i$. This criterion is similar to the one determined for the stability of a curved flame (Zel'dovich *et al.*

1980). In this later problem, the basic instability is the Darrieus–Landau hydrodynamic instability, the perturbations being advected by the tangential velocity field along the curved flame. The combination of these effects leads to total growth Γ proportional to the Reynolds number, which in some experiments can be large. Here, the basic instability is the thermoacoustic instability, whose growth rate, integrated on the residence time, leads to a total growth Γ proportional to the Zel'dovich number β, which is large.

6. Conclusions

In this paper we have provided a detailed study of a possible mechanism for the vibratory instability of a flame propagating in a tube. The driving mechanism results from the interaction between temperature variation in an acoustic wave and flame structure. The order of magnitude of the coupling constant is small, $\beta M \approx 10^{-2}$, but when integrated on the residential time, $t_{res} \approx \tau_a/M$, that the flame takes to propagate in the tube, the final amplification $\exp \Gamma$ of an initial perturbation can be very large because the Mach number disappears from the expression for Γ, which is found to be proportional to β. For sufficiently large Mach numbers, $M \geqslant 5\times10^{-3}$, the flame responds in a quasi-steady approximation $\tau_t < \tau_a$. In such cases, the numerical value of the proportionality coefficient which involves the specific heats of the reactive mixture and a global characteristic of the excited acoustic mode of the tube, is found from (5.12) to be small, of order 10^{-1}. Thus, in this case, the present instability mechanism does not lead to noticeable effects for typical values of the reduced activation energy ($10 \leqslant \beta \leqslant 20$). A strong vibratory instability is predicted to appear at sufficiently small Mach numbers, $M \approx 10^{-3}$, for which $\tau_t/\tau_a = O(1)$ and when a resonance phenomenon with the thermal–diffusive mechanisms is involved in the response of the flame structure. In the simplified flame model used here, this is the case for Lewis numbers sufficiently large, $\beta(Le-1) \geqslant 6$, which is representative of some lean mixtures of heavy hydrocarbons. In this case, islands of strong vibratory instability are predicted to appear in well-defined ranges of Mach numbers. A natural extension of this work would be to investigate the transfer of more complex flame models, including the effects of realistic chemical kinetics, to check the existence of such a strong vibratory instability for real flames.

Another important problem which remains is to evaluate the relative importance of all the possible mechanisms which can lead to a vibratory flame instability. As already mentioned in the introduction, two other mechanisms are possible. As suggested by Kaskan (1953), fluctuations of the heat released may result from the penetration of the flame edges into the acoustic boundary layers. As proposed by Markstein (1964), heat flux variations may also result from the variation of flame area by acceleration instability leading to unsteady cellular structures. Unfortunately, an accurate calculation of the transfer functions and of the stability limits of these last two mechanisms have not yet been carried out. By providing a detailed analysis of one of the possible mechanisms, the results presented in this paper appear as a contribution in the understanding of these vibratory instabilities.

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